Sankhyā : The Indian Journal of Statistics
1991, Volume **53**, Series A, Pt. 2, pp. 183-193.

A RANK TEST BASED ON THE NUMBER OF "NEAR-MATCHES" FOR ORDERED ALTERNATIVES IN RANDOMIZED BLOCKS

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SUMMARY. A rank test based on the number of "near-matches" among within-block rankings is proposed for ordered alternatives in a randomized block design with n treatments and m blocks. The Pitman asymptotic relative efficiency of this test with respect to the W-test based on within-block rankings is studied for uniform (or Wilcoxon) scores when m is fixed and the number of treatments n, is allowed to go to infinity. The relative performance of the proposed test and the W-test is studied for a number of situations involving the normal as well as some other heavy-tailed distributions. Monte Carlo studies have also been made for moderate values of n and m. Tables of critical values are provided for the proposed test for comparison of up to $n = 9$ treatments.

1. Introduction

Suppose $\{X_{ij}, j = 1, \ldots, n ; i = 1, \ldots, m\}$ represent the data in a complete Randomized Block Design experiment with n treatments and m blocks. Assume that the observation X_{ij} on the j-th treatment in the i-th block has the unknown cumulative distribution function F_{ij}. We restrict attention to ordered location shifts of the form $F_{ij}(t) = F(t-d_j-b_i)$ where $\{d_j\}$ are the treatment effects and $\{b_i\}$ are the (nuisance) block effects. We wish to test

$$H_0 : d_j = 0 \quad \forall j \qquad \ldots \quad (1.1)$$

versus the ordered alternative

$$F_1 : d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n, \qquad \ldots \quad (1.2)$$

where at least one of the inequalities in (1.2) is strict. We may rewrite this model as $X_{ij} = Y_{ij}+d_j+b_i$, $j = 1, \ldots, n ; i = 1, \ldots, m$ where $\{Y_{ij}\}$ are

Paper received. July 1988 ; revised September 1989.

AMS (1980) *subject classification.* 62G20, 62E20.

Key words. Rank test ; ordered alternatives ; randomized blocks ; near-matches ; asymptotic distribution ; asymptotic relative efficiencies.

independent identically distributed (i.i.d.) random variables (r.v.'s) having the common distribution function F with density f. Most of the parametric as well as nonparametric tests for the ordered alternatives in (1.2) have been proposed rather on an adhoc basis and they retain their (asymptotic or local) optimality properties only for nearly equal spacings of the d_j's i.e., the so-called regression alternatives. They may behave quite poorly for significant unevenness of these spacings. The tests based on isotonic maximum likelihood estimators have the property of being the most stringent and somewhere most powerful tests (see, for example, Schaafsma, 1968). However, these have only been worked out for the normal distributional models. In the nonparametric case the union-intersection (u.i.) principle has been used to construct such locally most powerful (L.M.P.) tests for ordered alternatives (see, for example, Chatterjee, 1984). These u.i. L.M.P. test statistics have rather complicated distributions in the null and specially in the alternative cases. The proposed test combines the simplicity of the null as well as asymptotic distribution theory while retaining the near optimality in a broader class of alternatives.

Let R_{ij} denote the rank of X_{ij} among $\{X_{i1}, \ldots, X_{in}\}$. Under H_0, $\boldsymbol{R}_i = (R_{i1}, \ldots, R_{in})$ takes on all possible $(n\,!)$ permutations of $\boldsymbol{n} = (1, \ldots, n)$, whereas under H_1, $\boldsymbol{R}_i$ is stochastically in the natural order. Therefore, it is natural to compare the concordance of the rankings within the i-th block, with the natural order. A test of this type, called the W-test, proposed and studied by Page (1963) is given by

$$W = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(j - \frac{n+1}{2}\right)\left(R_{ij} - \frac{n+1}{2}\right). \qquad \ldots \quad (1.3)$$

The test rejects H_0 for large values of Pirie (1974) derived the asymptotic relative efficiency (ARE) of this test with respect to Hollander's (1966) for fixed m and $n \to \infty$. Also, Pirie and Hollander (1972) and Pirie (1985) considered the normal–scores version of this test. Observe that (1.3) can be expressed equivalently as $-\sum_{i=1}^{m} \|\boldsymbol{R}_i - \boldsymbol{n}\|^2$, where $\|.\|$ stands for the Euclidean norm. Incorporating the notion of nearest neighbor norm, i.e., $\rho_k(\|\boldsymbol{a}-\boldsymbol{b}\|) = \{\#j : |a_j - b_j| \leqslant k\}$ we propose an alternative test, designated as the M-test, based on the statistic given by

$$M = \sum_{i=1}^{m} \sum_{j=1}^{n} I(|R_{ij} - j| \leqslant k_{j,n})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} I\left(b_{nj} \leqslant \left(\frac{R_{ij}}{n}\right) \leqslant a_{nj}\right) \qquad \ldots \quad (1.4)$$

with

$$a_{nj} = n^{-1} \min(n, j+k_{j,n})\, b_{nj} = n^{-1} \max(0, j-k_{j,n}) \qquad \dots \quad (1.5)$$

and where $I(A)$ is the usual indicator function of the event A. Note that a perfect match relates to the case of $k = 0$ while for $k \geqslant 1$ (but small compared to n) we have "near matches". For the particular case of $m = 1$ and $k_{j,n}$ independent of j we may refer to Jammalamadaka and Janson (1984). The asymptotic distribution of the M statistic under a converging sequence of alternatives given in (2.2), is derived in Section 2 while the ARE of the M-test with respect to the W-test is considered in Section 3. In Section 4, Monte Carlo power computations for the M and W tests, for $n = 8$ and $m = 2$ are given. Also, the tables of critical values of M-test statistic, for values of $n = 3(1)9$, $m = 1(1)4$ for nominal values of $\alpha = .10\ .05$ and .01 are given.

In passing, we may mention that for moderate values of n, Pearsonian distributional approximations can be adopted through computed values of skewness and kurtosis coefficients under the null hypothesis.

2. The asymptotic distribution of the M statistic

For ARE computations, consider a Pitman-sequence $\{H_{1n}\}$ of local alternatives, where, subject to (1.2),

$$H_{1n} : X_{ij} = Y_{ij}+d_{nj}+b_i, j = 1, \dots, n\ ; i = 1, \dots, m\ ; \qquad \dots \quad (2.1)$$

with

$$\max\{|d_{nj}| : 1 \leqslant j \leqslant n\} = O(n^{-1/2}). \qquad \dots \quad (2.2)$$

It is clear that statistics based on within-block rankings (as our M-test), eliminate the (nuisance) block effects. Note that the sequence of alternatives in (2.2) could also have been defined with $d_{nj} = \varphi_n . d_j$ where $\varphi_n = n^{-1/2}\ \varphi$. We emphasize again that the d_{nj}'s in (2.1) need not be equally spaced.

Let $F(y)$ be the true d.f. of Y_{11} so that under H_{1n}, the d.f. of X_{ij} is $F(y-d_{nj}), j = 1, \dots, n$. Let $F_{ni}(y) = n^{-1} \sum_{j=1}^{n} I(X_{ij} \leqslant y)$ be the empirical d.f. of $X_{i1}, \dots, X_{in}$. Then, the M statistic in (1.4) can be expressed as $\sum_{i=1}^{m} \sum_{j=1}^{n} I(|nF_{ni}(X_{ij})-j| \leqslant k_{j,n})$. Also, let $\bar{F}_{(n)}(y) = n^{-1} \sum_{j=1}^{n} F(y-d_{nj})$ be the average d.f.. Suppose we have the

Assumption (A) : F has a bounded (a.e.) second derivative $f'(=F'')$. Then, by a Taylor expansion and (2.2), we have, for $0<\theta<1$,

$$\bar{F}_{(n)}(y) = n^{-1}\sum_{j=1}^{n} F(y-d_{nj}) = n^{-1}\sum_{j=1}^{n}\left\{F(y)-d_{nj}f(y)+\frac{1}{2}\,d_{nj}^2 f'(y-\theta d_{nj})\right\}$$

$$= F(y)+(2n)^{-1}\sum_{j=1}^{n} d_{nj}^2 f'(y-\theta d_{nj}).$$

Hence,

$$\sup_y\ |\bar{F}_{(n)}(y)-F(y)| \leqslant (2n)^{-1}\{\sup_y |f'(y)|\} = O(n^{-1}). \qquad \ldots \quad (2.3)$$

Note that in this case of independent but non-identically distributed r.v.'s, the tightness part of the weak convergence of the empirical process ensures the following (for each i) :

$$n^{1/2}\|F_{ni}-F\| = \sup_y\ \{n^{1/2}|F_{ni}(y)-F(y)|\} = O_p(1) \qquad \ldots \quad (2.4)$$

and, for every $\epsilon>0$, there exists a positive δ, such that (cf. Shorack and Wellner, 1986, p. 109, Theorem 1)

$$\sup\{n^{1/2}|\bar{F}_{(n)}(y)-F(y)-F_{ni}(x)+F(x)| \ : \ |y-x|<\delta,\, x, y \in \boldsymbol{R}\} \leqslant \epsilon. \quad \ldots \quad (2.5)$$

Let $k_{j,n}$ appearing in (1.4) be defined by $n^{-1}k_{j,n} = k_n\left(\frac{j}{n+1}\right), j=1, \ldots, n,$ such that (i) $k_n(u)$ is constant on the interval $\left(\frac{j-1}{n+1}, \frac{j}{n+1}\right], j=1, \ldots, n+1$, and (ii) for each $u\in(0,1)$, $k_n(u)$ converges as $n\to\infty$, to $k(u)$, which we assume to be continuous except for finitely many u, bounded below by zero and bounded above by an integrable function. These assumptions imply that $0<\int 2k(u)[1-2k(u)]du<\infty$. Then we have the following

Theorem 2.1. *Under* $\{H_{1n}\}$ *in* (2.1), (2.2) *and Assumption* (A),

$[M-m(\mu_n^0+\Delta_n)]/\sqrt{mn}\,\sigma \xrightarrow{d} N(0,1)$ *as* $n\to\infty$, *where* $\mu_n^0 = \sum_{j=1}^{n}(a_{nj}-b_{nj})$,

$$\Delta_n = -\sum_{j=1}^{n} d_{nj}[f(F^{-1}(a_{nj}))-f(F^{-1}(b_{nj}))], \qquad \ldots \quad (2.6)$$

and

$$\sigma^2 = \int_0^1 2k(u)[1-2k(u)]du. \qquad \ldots \quad (2.7)$$

Proof. Note that

$$M = \sum_{i=1}^{m} Z_{ni} = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} Z_{nij} \right\}, \qquad \ldots \quad (2.8)$$

where $Z_{ni} = \sum_{j=1}^{n} I(b_{nj} \leqslant F_{ni}(X_{ij}) \leqslant a_{nj})$, so that for each j,

$$\begin{aligned} Z_{nij} &= I[b_{nj} \leqslant F_n(X_{ij}) \leqslant a_{nj}] \\ &= I[b_{nj} - \{F_n(X_{ij}) - F(X_{ij})\} \leqslant F(X_{ij}) \leqslant a_{nj} - \{F_{n,i}(X_{ij}) - F(X_{ij})\}] \\ &= I[F^{-1}(b_{nj} - \{F_{ni}(X_{ij}) - F(X_{ij})\}) - d_{nj} \leqslant Y_{ij} \\ &\leqslant F^{-1}(a_{nj} - \{F_{n,i}(X_{ij}) - F(X_{ij})\}) - d_{nj}]. \end{aligned} \qquad \ldots \quad (2.9)$$

Writing

$$Z^0_{nij} = I[F^{-1}(b_{nj}) - d_{nj} \leqslant Y_{ij} \leqslant F^{-1}(a_{nj}) - d_{nj}],\ j = 1, \ldots, n\ ; \qquad \ldots \quad (2.10)$$

and

$$Z^0_{ni} = \sum_{j=1}^{n} Z^0_{nij}, \qquad \ldots \quad (2.11)$$

we have from the above that

$$Z_{ni} - Z^0_{ni} = \sum_{j=1}^{n} [Z_{nij} - Z^0_{nij}]. \qquad \ldots \quad (2.12)$$

By using (2.3) and (2.4) and the definitions of Z_{nij} and Z^0_{nij} [see (2.9) and (2.10)], we conclude that the interval (in y) over which Z_{nij} and Z^0_{nij} have the same value differ by a random shift of the order of $O_p(n^{-1/2})$. Using (2.5) and the fact (cf. Shorack and Wellner, 1986, p. 109, Theorem 1) that $\sqrt{n}(F_{n,i}(x) - F(x))$ has asymptotically a normal distribution with zero mean and a finite variance ($\leqslant 1/4$), we can conclude by some standard analysis that $P\{Z_{nij} - Z^0_{nij} = +1\} = o(n^{-1/2}) = P\{Z_{nij} - Z^0_{nij} = -1\}$, and $P\{Z_{nij} - Z^0_{nij} = 0\} = 1 - o(n^{-1/2})$. These yield that $E\,|\,Z_{nij} - Z^0_{uij}\,| = o(n^{-1/2})$, $\forall\, j$ which, in view of (2.12), implies that $n^{-1/2}\, E\,|\,Z_{ni} - Z^0_{ni}\,| = o(1)$, and hence $n^{-1/2}\,(Z_{ni} - Z^0_{ni}) \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, using Taylor expansions, one can see that the Z^0_{nij}'s are independent r.v.'s with

$$EZ^0_{nij} = (a_{nj} - b_{nj}) - d_{nj}[f(F^{-1}(a_{nj})) - f(F^{-1}(b_{nj}))] + O(n^{-1}), \qquad \ldots \quad (2.13)$$

$$\operatorname{var}(Z^0_{nij}) = (a_{nj} - b_{nj})\,[1 - (a_{nj} - b_{nj})] + O(n^{-1/2}). \qquad \ldots \quad (2.14)$$

Thus, by (2.11), (2.13) and (2.14), we have

$$E(Z^0_{ni}) = \mu^0_n + \Delta_n + O(1), \qquad \ldots \quad (2.15)$$

$$\operatorname{var}(Z^0_{ni}) = \sum_{j=1}^{n} (a_{nj} - b_{nj})[1 - (a_{nj} - b_{nj})] + O(n^{-1/2}).$$

Observe that in the expression for $E(Z^0_{ni})$, it may be checked that $\mu^0_n \sim n$ and $\Delta_n \sim \sqrt{n}$ [see (2.18)] while

$$n^{-1}.\ V(Z^0_{ni}) \sim n^{-1} \sum_{j=1}^{n} n^{-1}(2k_{j,n}+1)[1-n^{-1}(2k_{j,n}+1)]$$

$$\sim n^{-1} \sum_{j=1}^{n} 2k_n\left(\frac{j}{n+1}\right)\left[1-2k_n\left(\frac{j}{n+1}\right)\right]$$

$$\to \sigma^2 = \int_0^1 2k(u)[1-2k(u)]du, \text{ as } n \to \infty. \qquad \ldots \quad (2.16)$$

Since the Z^0_{nij}'s are bounded and independent, using the classical central limit theorem for triangular arrays, we conclude that

$$n^{-1/2}\{Z^0_{ni}-[\mu^0_n+\Delta_n]\}/\sigma \xrightarrow{d} N(0, 1) \text{ as } n \to \infty.$$

This implies, in turn that $n^{-1/2}\{Z_{ni}-[\mu^0_n+\Delta_n]\}/\sigma \xrightarrow{d} N(0, 1)$ as $n \to \infty$. Now, using (2.8) and the fact that Z_{ni}, $i = 1, \ldots, m$ are i.i.d., we obtain that $\{M-m[\mu^0_n+\Delta_n]\}/\sqrt{mn}\,\sigma \xrightarrow{d} N(0, 1)$ as $n \to \infty$. □

From (2.6) note that

$$\Delta_n = \sum_{j=1}^{n} d_{nj}[fF^{-1}(b_{nj}))-f(F^{-1}(a_{nj}))]$$

$$\sim \sum_{j=1}^{n} d_{nj}(a_{nj}-b_{nj})\left\{-f'\left(F^{-1}\left(\frac{j}{n+1}\right)\right)\Big/f\left(F^{-1}\left(\frac{j}{n+1}\right)\right)\right\}+o(1).$$

Hence, from (2.16)

$$\Delta_n/\sigma \sim \frac{2\sum_{j=1}^{n} d_{nj}k_n\left(\frac{j}{n+1}\right)\left\{-f'\left(F^{-1}\left(\frac{j}{n+1}\right)\right)\Big/f\left(F^{-1}\left(\frac{j}{n+1}\right)\right)\right\}}{\left\{\int_0^1 2k(u)[1-2k(u)]du\right\}^{1/2}}+o(1) \qquad \ldots \quad (2.17)$$

Thus, by Assumption (A) and (2.2) if the density f is sufficiently regular viz., strongly unimodal (cf. Hájek and Sidák 1967, p. 15), the r.h.s. of (2.17) is finite and converges to a limit (as $n \to \infty$) under fairly general conditions. Finally let $\psi(u) = -\{f'(F^{-1}(u))/f(F^{-1}(u))\}$, $0 < u < 1$ be the usual score function and assume

Assumption (B). $\psi(u)$ is continuous except possibly for finitely many u and is bounded in absolute value by an integrable function.

Let $d_{nj} = n^{-1/2}d\left(\frac{j}{n+1}\right)$, $j = 1, \ldots, n$, for some monotone function $d(.) = \{d(u),, 0 < u < 1\}$ which may have finitely many jumps. This includes, for example, change-point alternatives of the form $d_1 \leqslant d_2 = d_3 \leqslant d_4$. Then, from (2.17), we obtain that as $n \to \infty$

$$n^{-1/2}\,\Delta_n/\sigma \to \sqrt{2}\int_0^1 d(u)k(u)\psi(u)du \Big/ \left\{\int_0^1 k(u)[1-2k(u)]du\right\}^{1/2} \qquad \dots \quad (2.18)$$

Thus, we have proved the following

Corollary 2.2 : *Under* (2.2) *and Assumptions* (A) *and* (B), $[M-m\mu_n^0]/\sqrt{mn}\,\sigma \xrightarrow{d} N\left(\sqrt{2}m\int_0^1 d(u)k(u)\psi(u)du \Big/ \left\{\int_0^1 k(u)[1-2k(u)]du\right\}^{1/2}, 1\right)$ as $n \to \infty$. □

Consider the special case of $k = k_{j,n}$ (independent of j) where a match is defined when $j-k \leqslant R_{ij} \leqslant j+k$. Then we have (cf.(1.5)) $a_{nj} = n^{-1}\min(n, j+k)$ and $b_{nj} = n^{-1}\max(0, j-k)$. If $k/n \to p(0 < p < 1/2)$, then the limiting window function, $\lim_n (a_{nj}-b_{nj})/2$, corresponds to

$$k(u) = \begin{cases} \tfrac{1}{2}(u+p), & \text{for } 0 < u \leqslant p \\ p &, \ p \leqslant u \leqslant 1-p \\ \tfrac{1}{2}(1-u+p) & 1-p \leqslant u < 1\,; \end{cases} \qquad \dots \quad (2.19)$$

for which $\sigma_p^2 = \int_0^1 2k(u)[1-2k(u)]du = 2p(1-p)\ (1-2p)$. Theorem 2.1 yields the following

Corollary 2.3. *Under the hypothesis of Theorem* 2.1, *for* $k_{n,j} = k_n = [np]$, $0 < p < 1/2$, $\{M-m[(2k_n+1)+\Delta_n]\}/\sqrt{mn}\,\sigma_p \xrightarrow{d} N(0, 1)$ *as* $n \to \infty$. *Further, if f satisfies Assumption* (B), $(M-m(2k_n+1)]]/\sqrt{mn}\,\sigma_p \xrightarrow{d} N(\sqrt{2m}\int_0^1 d(u)k(u)\psi(u)du/\sigma_p, 1)$ *as* $n \to \infty$.

Remark 2.4. An approximate α-level test is given by : reject H_0 in favor of H_1 if $[\{M-m[\mu_n^0+\Delta_n]\}/\sqrt{mn}\,\sigma] \geqslant z_\alpha$, where z_α is the $100(1-\alpha)$-th percentile of a standard normal distribution.

3. Asymptotic relative efficiencies

Consider the sequence $\{H_{1n}\}$ of alternatives in (2.1) with $d_{nj} = \sqrt{12}\left(j-\frac{n+1}{2}\right)\Big/\sqrt{n(n^2-1)}$ which corresponds to $d(u) = \sqrt{12}(u-1/2)$ or to $d_j \equiv j/n$ in (1.2). Then

$$\begin{aligned} E\{W \mid H_{1n}\} &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(j-\frac{n+1}{2}\right)E\,(R_{ij} \mid H_{1n}) \\ &= m\sum_{j=1}^{n}\left(j-\frac{n+1}{2}\right)\left[1+\sum_{\substack{l=1\\ \neq j}}^{n} E\{\mathrm{I}(X_{il} < X_{ij}) \mid H_{1n}\}\right] \\ &= m\sum_{j=1}^{n}\left(j-\frac{n+1}{2}\right)\left[\frac{n+1}{2}+nd_{nj}(f^2(x)dx+o(n^{-1/2})\right] \\ &= mn\,(\textstyle\int f^2(x)dx)\,\sqrt{n(n^2-1)/12}+o(n^{-1/2}). \end{aligned}$$

Also (cf. Pirie, 1974, equation (3.13)) $\mathrm{var}(W \mid H_0) = \dfrac{mn^2(n+1)^2(n-1)}{144}$. Since $E(W \mid H_0) = 0$, the "efficacy" of W under $\{H_{1n}\}$ is

$$\lim_{n \to \infty} \{m^2n^2 (\int f^2(x)dx)^2 \, n(n^2-1)/12]/[mn^2(n+1)^2(n-1)/144]\}$$
$$= 12\, m\, (\int f^2(x)dx)^2. \qquad \ldots \quad (3.1)$$

Likewise from (2.18) the efficacy of M under $\{H_{1n}\}$ is

$$(2m)\left\{ \int_0^1 d(u)\, \psi(u)\, k(u)du \right]^2 \Big/ \left\{ \int_0^1 k(u)[1-2k(u)]du \right\}. \qquad \ldots \quad (3.2)$$

From (3.1) and (3.2) the ARE (cf. Hájek and Sidak 1967, p. 267) is

$$e(M, W) = \left[\int_0^1 d(u)\, \psi(u)\, k(u)du \right]^2 \Big/ \left\{ 6\left(\int f^2(x)dx)^2 \left(\int_0^1 k(u)[1-2k(u)]du \right) \right\}.$$

$$= 2\left[\int_0^1 (u-1/2)\psi(u)k(u)du \right]^2 \Big/ \left\{ \left(\int f^2(x)dx)^2 \left(\int_0^1 k(u)[1-2k(u)]du \right) \right\}.$$
$$\ldots \quad (3.3)$$

Let the window-function $k(u)$ be given by (2.19). For the normal, logistic, double exponential and Cauchy distributions the values of the asymptotic relative efficiency, in equation (3.3) are evaluated and given in Table 1 below. These calculations show that for all these distributions, $e(M, W)$ exceeds one for values of p in the range of (.35,, .40) with higher efficiency nearer 0.4.

TABLE 1. THE RELATIVE EFFICIENCY $e(M, W)$ OF THE MATCH TEST AGAINST PAGE'S TEST

	normal	logistic	double exponential	cauchy
$p = 0.30$	0.9728	0.7882	0.1400	1.4476
$p = 0.35$	1.5519	2.4168	1.0393	2.2125
$p = 0.40$	2.3766	7.0052	4.0563	3.7761

From the above examples, it is clear that the optimal choice of the window length k_n depends on the underlying density (through $\psi(u)$) as well as, on the alternatives (through $d(u)$). No such choice may remain optimal for a given density, for different alternatives $d(u)$ or vice-versa. This question is being further investigated. We may also remark that the heavier the tails of the distribution from which the observations come, the smaller the window length to achieve this superior performance of the match test. While these are asymptotic results, the Monte Carlo results at the end of this section as well as other empirical powers we have generated, speak very favorably of the match test M.

4. Tables, illustration and some Monte Carlo power computations

Table 2 provides the critical values of the M statistic for values of $n = 3(1)9$, $m = 1(1)4$ for nominal values of $\alpha = 0.10, 0.05, 0.01$. Since the distribution of M is discrete, corresponding to these critical values the exact probabilities of exceeding, are given in parentheses. For example, if $n = 6$, $m = 4$ using $k = 2$ and $\alpha = 0.01$, the critical values of 21 (0.006) and 20 (0.022) bracket the α value of 0.01.

TABLE 2. CRITICAL VALUES AND EXACT PROBABILITIES FOR THE M-STATISTIC

n	$/\alpha \rightarrow$	$m = 1$ .10	.05	.01	$m = 2$ .10	.05	.01
3	$k = 1$	3(.000)	3(.000)	3(.000)	6(.000)	6(.000)	6(.000)
		2(.500)	2(.500)	2(.500)	5(.250)	5(.250)	5(.250)
4	$k = 1$	4(.000)	4(.000)	4(.000)	5(.043)	7(.043)	8(.000)
		3(.208)	3(.208)	3(.208)	6(.148)	6(.148)	7(.043)
5	$k = 1$	4(.067)	5(.000)	5(.000)	7(.089)	8(.022)	9(.004)
		3(.200)	4(.067)	4(.067)	6(.218)	7(.089)	8(.022)
	$k = 2$	5(.000)	5(.000)	5(.000)	9(.067)	10(.000)	10(.000)
		4(.258)	4(.258)	4(.258)	8(.256)	9(.067)	9(.067)
6	$k = 1$	4(.068)	5(.018)	6(.000)	8(.042)	8(.042)	10(.002)
		3(.278)	4(.068)	5(.018)	7(.122)	7(.122)	9(.012)
	$k = 2$	6(.000)	6(.000)	6(.000)	10(.056)	11(.010)	12(.000)
		5(.101)	5(.101)	5(.101)	9(.181)	10(.056)	11(.010)
7	$k = 1$	5(.019)	5(.019)	6(.004)	8(.059)	9(.020)	10(.005)
		4(.106)	4(.106)	5(.019)	7(.144)	8(.059)	9(.020)
	$k = 2$	6(.034)	6(.034)	7(.000)	11(.039)	11(.039)	12(.009)
		5(.144)	5(.144)	6(.034)	10(.116)	10(.116)	11(.039)
	$k = 3$	7(.000)	7(.000)	7(.000)	12(.098)	13(.018)	14(.000)
		6(.134)	6(.134)	6(.134)	11(.279)	12(.098)	13(.018)
8	$k = 1$	5(.033)	5(.033)	6(.005)	8(.071)	9(.027)	10(.008)
		4(.110)	4(.110)	5(.033)	7(.159)	8(.071)	9(.027)
	$k = 2$	6(.053)	7(.010)	7(.010)	11(.070)	12(.023)	13(.006)
		5(.200)	6(.053)	6(.053)	10(.163)	11(.070)	12(.023)
	$k = 3$	7(.051)	8(.000)	8(.000)	13(.077)	14(.019)	16(.000)
		6(.215)	7(.051)	7(.051)	12(.203)	13(.077)	15(.003)
9	$k = 1$	5(.036)	5(.036)	6(.009)	8(.081)	9(.032)	11(.003)
		4(.122)	4(.122)	5(.036)	7(.171)	8(.081)	10(.011)
	$k = 2$	6(.082)	7(.017)	8(.003)	11(.096)	12(.038)	14(.003)
		5(.220)	6(.082)	7(.017)	10(.199)	11(.096)	13(.013)
	$k = 3$	7(.093)	8(.018)	9(.000)	14(.054)	15(.016)	16(.003)
		6(.288)	7(.093)	8(.018)	13(.140)	14(.054)	15(.016)
	$k = 4$	8(.070)	9(.000)	9(.000)	16(.034)	16(.034)	17(.005)
		7(.282)	8(.070)	8(.070)	15(.125)	15(.125)	16(.034)

	m = 3			m = 4		
n /α→	*.10	.05	.01	.10	.05	.01
3 k = 1	9(.000)	9(.000)	9(.000)	11(.063)	12(.000)	12(.000)
	8(.125)	8(.125)	8(.125)	10(.229)	11(.063)	11(.063)
4 k = 1	10(.042)	10(.042)	11(.009)	13(.042)	13(.042)	15(.002)
	9(.135)	9(.135)	10(.042)	12(.113)	12(.113)	14(.011)
5 k = 1	10(.097)	11(.046)	13(.002)	13(.098)	14(.043)	16(.005)
	9(.206)	10(.097)	12(.011)	12(.191)	13(.098)	15(.016)
k = 2	13(.091)	14(.017)	15(.000)	18(.030)	18(.030)	19(.004)
	12(.255)	13(.091)	14(.017)	17(.104)	17(.104)	18(.030)
6 k = 1	11(.061)	12(.023)	13(.008)	14(.071)	15(.032)	17(.004)
	10(.134)	11(.061)	12(.023)	13(.138)	14(.071)	16(.013)
k = 2	15(.035)	15(.035)	16(.008)	19(.061)	20(.022)	21(.006)
	14(.103)	14(.103)	15(.035)	18(.139)	19(.061)	20(.022)
7 k = 1	11(.081)	12(.035)	14(.004)	14(.093)	15(.047)	17(.009)
	10(.160)	11(.081)	13(.013)	13(.167)	14(.093)	16(.022)
k = 2	15(.088)	16(.035)	18(.003)	20(.066)	21(.029)	23(.003)
	14(.183)	15(.085)	17(.011)	19(.133)	20(.066)	22(.010)
k = 3	18(.073)	19(.018)	20(.002)	24(.054)	25(.016)	26(.003)
	17(.191)	18(.073)	19(.018)	23(.135)	24(.054)	25(.016)
8 k = 1	11(.096)	12(.045)	14(.007)	15(.059)	16(.029)	18(.005)
	10(.179)	11(.096)	13(.019)	14(.110)	15(.059)	17(.013)
k = 2	16(.066)	17(.027)	18(.010)	21(.059)	22(.027)	24(.004)
	15(.135)	26(.066)	17(.027)	20(.112)	21(.059)	23(.011)
k = 3	19(.083)	20(.030)	21(.008)	25(.081)	26(.034)	28(.003)
	18(.182)	19(.082)	20(.030)	24(.162)	25(.081)	27(.012)
9 k = 1	12(.054)	13(.024)	14(.010)	15(.069)	16(.036)	18(.007)
	11(.107)	12(.054)	13(.024)	14(.124)	15(.069)	17(.017)
k = 2	26(.094)	17(.045)	19(.007)	21(.088)	22(.046)	24(.010)
	15(.174)	16(.094)	18(.019)	20(.151)	21(.088)	23(.022)
k = 3	20(.076)	21(.031)	23(.003)	26(.087)	27(.042)	29(.007)
	19(.155)	20(.076)	22(.010)	25(.160)	26(.087)	28(.018)
k = 4	23(.059)	24(.018)	25(.003)	30(.067)	31(.030)	32(.009)
	22(. 49)	23(.059)	24(.018)	29(.161)	30(.076)	31(.030)

Table 3 gives Monte Carlo powers of M and W for $n=8$ and $m=2$ for normal data based on 1,000 replications for each value of power. We consider three different alternatlves, H_r : $d_j=0.1+j(0.1)r$, $j=0,\ldots 7$; $r=1,2,3$. The table shows that the power of M is considerably better than W and that it increases with p.

TABLE 3. MONOTE CARLO POWER COMPARISONS OF M AND PAGE'S W-TEST STATISTICS FOR $n = 8$ AND $m = 2$ ($\alpha = 0.05$)

alternatives	M test $p = 1/8$	1/4	1/3	W test
H_1	0.333	0.382	0.448	0.227
H_2	0.524	0.627	0.738	0.505
H_3	0.749	0.839	0.906	0.788

Acknowledgement. The authors would like to thank the referees for their valuable comments. These tables of critical values and powers were computed at the University of California, Santa Barbara, on IBM 4341 (CPU) VMSP and we acknowledge the assistance of Mr. Mohammad Kafai. Additional tables of critical values or the computer program may be obtained from the authors.

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